Structure of the Set of Belief Functions Generated by a Random Closed Interval

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***Abstract*—Geometrical and topological properties of the set of all belief functions generated by a random closed interval are studied. It is shown that this set is a metrizable (noncompact) simplex and its extreme points are completely characterized.**

Keywords: belief function, random closed interval, simplex, extreme point

I. Introduction

The study of belief measures (functions) on infinite universes was initiated by Dempster. In his paper [1] the upper and lower probabilities are generated by a multivalued mapping and a rule for combining sources of information is proposed. In particular, this multivalued mapping takes the form of a random closed interval in the subsequent paper [2]. This approach enables assigning degrees of belief and plausibility to subsets of real numbers as in the the usual finite framework for Dempster-Shafer theory [3]. Smets further investigated the belief functions on reals in [4] and their application to data analysis and uncertain knowledge representation is shown in the recent articles [5], [6].

The aim of this contribution is to study the geometrical-topological structure of the set $\mathbf{Bel}$ of all belief functions on reals. By establishing a one-to-one correspondence between the simplex of certain Borel probability measures and $\mathbf{Bel}$ we will show that $\mathbf{Bel}$ is a simplex whose extreme points can be fully described. Several published papers already dealt with the similar topic. Shafer studied in [7] belief functions on a system of sets of an infinite universe and their representation by a probability measure or a probability charge. Denneberg and Brüning [8] identified the extreme points of the compact convex set of all belief functions on an algebra of sets. The approach pursued herein can be put in contrast with the above mentioned papers in certain points:

(i) We follow the M.H. Stone's maxim "One must always topologize" so that the symbiosis between the measure-theoretic structure and topology is emphasized from the beginning.
(ii) The set of all belief functions $\mathbf{Bel}$ is not compact.
(iii) The total monotonicity of a belief function is rather a derived concept and not a primitive one (in accordance with [2]).

It should be mentioned that investigation of the convex set of totally monotone capacities on a topological space appeared already in the Choquet's foundational work [9]. While some results proved in this paper are implicitly stated in the Choquet's paper, we provide a direct proof following the latest development of topological capacities [10].

This article is structured as follows. In the next section we introduce basic notions and definitions concerning belief functions and geometry of simplices. Section III contains main results (Proposition 3 and Theorem 2) and in Section IV we compare the results obtained in this article with the properties of belief functions on other universes such as the set of all subsets of a finite set or an algebra of sets.

II. Basic Notions

A. Belief measures on real numbers

If $S \subseteq \mathbb{R}^n$, then $\mathcal{B}(S)$ denotes the Borel subsets of S. By $\mathcal{K}$ and $\mathcal{G}$ we denote the set of all compact subsets and the set of all open subsets of $\mathbb{R}$, respectively.

Let P be a probability distribution of a real random vector (X, Y) with $P[X \leq Y] = 1$. Such a probability distribution is just a Borel probability measure on $\mathcal{B}(\mathbb{R}^2)$ whose *support*

$$\operatorname{spt} P = \mathbb{R}^2 \setminus \bigcup\{G \mid G \subseteq \mathbb{R}^2, G \text{ open}, P(G) = 0\}$$

is included in the halfplane

$$H = \{(x, y) \in \mathbb{R}^2 \mid x \leq y\}.$$

Note that we always consider H equipped with the subspace Euclidean topology of $\mathbb{R}^2$. The Borel probability measure P is automatically *inner regular* with respect to the compact subsets of H, that is, the equality

$$P(A) = \sup\{P(K) \mid K \subseteq A, K \text{ compact}\},$$

holds true for every $A \in \mathcal{B}(H)$.

In the sequel, we use for every $A \in \mathcal{B}(\mathbb{R})$ the expressions

$$[X, Y] \subseteq A$$

and

$$[X, Y] \cap A \neq \emptyset$$

as shortcuts for

$$\{(x, y) \in H \mid [x, y] \subseteq A\}$$

and

$$\{(x,y) \in H \mid [x,y] \cap A \neq \emptyset\},$$

respectively. In the light of [11], we call $[X,Y]$ a *random closed interval*.

The following definition has its origin in [2]. The probability distribution P of the random vector (X,Y) plays exactly the same role as a basic (mass) assignment in the finite setting for Dempster-Shafer theory.

Definition 1. *A* belief measure *on* $\mathcal{B}(\mathbb{R})$ *is a function*

$$\mathrm{Bel} : \mathcal{B}(\mathbb{R}) \to [0,1]$$

such that

$$\mathrm{Bel}(A) = P[[X,Y] \subseteq A], \quad A \in \mathcal{B}(\mathbb{R}). \tag{1}$$

The function $\mathrm{Pl} : \mathcal{B}(\mathbb{R}) \to [0,1]$ *defined by*

$$\mathrm{Pl}(A) = P[[X,Y] \cap A \neq \emptyset], \quad A \in \mathcal{B}(\mathbb{R}), \tag{2}$$

is called a plausibility measure.

Observe that

$$\mathrm{Bel}(A) = 1 - \mathrm{Pl}(\overline{A}),$$

where $A \in \mathcal{B}(\mathbb{R})$ and $\overline{A}$ denotes the complement of A. Because of the obvious duality between belief measures and plausibility measures, we will confine to the investigation of belief measures.

The following properties of a belief measure are direct consequences of the definition (1) together with inner regularity of the Borel probability measure P.

Proposition 1. *Every belief measure* Bel *on* $\mathcal{B}(\mathbb{R})$ *has the following properties:*

(i) $\mathrm{Bel}(\emptyset) = 0, \mathrm{Bel}(\mathbb{R}) = 1$

(ii) if $A, B \in \mathcal{B}(\mathbb{R})$ *are such that* $A \subseteq B$*, then*

$$\mathrm{Bel}(A) \leq \mathrm{Bel}(B)$$

(iii) if $n \geq 2$ *and* $A_1, \dots, A_n \in \mathcal{B}(\mathbb{R})$*, then*

$$\mathrm{Bel}\Big(\bigcup_{i=1}^{n} A_i\Big) + \sum_{\substack{I \subseteq \{1,\dots,n\} \\ I \neq \emptyset}} (-1)^{|I|} \mathrm{Bel}\Big(\bigcap_{i \in I} A_i\Big) \geq 0$$

(iv) if $(A_n) \in \mathcal{B}(\mathbb{R})^{\mathbb{N}}$*, where* $A_n \supseteq A_{n+1}$ *for each* $n \in \mathbb{N}$*, then*

$$\lim_{n \to \infty} \mathrm{Bel}(A_n) = \mathrm{Bel}\Big(\bigcap_{n=1}^{\infty} A_n\Big)$$

(v) if $(A_n) \in \mathcal{B}(\mathbb{R})^{\mathbb{N}}$*, where* $A_n \subseteq A_{n+1}$ *for each* $n \in \mathbb{N}$*, then*

$$\lim_{n \to \infty} \mathrm{Bel}(A_n) = \mathrm{Bel}\Big(\bigcup_{n=1}^{\infty} A_n\Big)$$

(vi) $\mathrm{Bel}(A) = \sup\{\mathrm{Bel}(K) \mid K \subseteq A, K \in \mathcal{K}\}$*, for every* $A \in \mathcal{B}(\mathbb{R})$

In particular, the properties (ii)-(iii) say that every belief function Bel is a *totally monotone* set function on $\mathcal{B}(\mathbb{R})$.

Every belief measure on $\mathcal{B}(\mathbb{R})$ can be extended to a so-called capacity defined on the set $2^{\mathbb{R}}$ of all subsets of $\mathbb{R}$. We adopt the definition of capacity used in [10, Definition 1.1]. A *capacity* is a function $c : 2^{\mathbb{R}} \to [0,\infty]$ such that

(i) $c(\emptyset) = 0$,

(ii) if $A \subseteq \mathbb{R}$, then $c(A) = \sup\{c(K) \mid K \subseteq A, K \in \mathcal{K}\}$,

(iii) if $K \in \mathcal{K}$, then $c(K) = \inf\{c(G) \mid G \supseteq K, G \in \mathcal{G}\}$.

Every capacity is thus uniquely determined by its values on $\mathcal{K}$. Since every belief measure Bel is continuous from above by (iv) in Proposition 1, it results from [10, Proposition 1.1] that the set function Bel' defined by

$$\mathrm{Bel}'(A) = \sup\{\mathrm{Bel}(K) \mid K \subseteq A, K \in \mathcal{K}\}, \quad A \subseteq \mathbb{R},$$

is a capacity. Hence we may identify each belief measure Bel with the restriction to $\mathcal{B}(\mathbb{R})$ of the capacity Bel'. We will tacitly assume this identification in Section III.

B. Convex sets and simplices

There appear more different notions of a simplex in the literature so we repeat all the necessary definitions for the sake of clarity. Our exposition is based on [12], where the original Choquet's definition of simplex is used.

Let E be a real linear space. A *convex set* in E is any subset K of E that is closed under *convex combinations:* if $x_1, \dots, x_n \in K$ and $\alpha_i \geq 0$ with $\sum_{i=1}^{n} \alpha_i = 1$, then

$$\alpha_1 x_1 + \dots + \alpha_n x_n \in K.$$

An *extreme point* of a convex set K is a point $e \in K$ such that the set $K \setminus \{e\}$ is convex. The set

$$\mathrm{ext}\, K = \{x \in K \mid x \text{ is an extreme point of } K\}$$

is called an *extreme boundary of* K. By an *affine combination* we mean a linear combination $\alpha_1 x_1 + \dots + \alpha_n x_n$ with

$$\alpha_1 + \dots + \alpha_n = 1, \ \ \alpha_i \in \mathbb{R}.$$

A subset A of E is said to be *affinely independent* if there does not exist an element $a \in A$ that can be expressed as an affine combination of elements from $A \setminus \{a\}$. An *affine subspace* of E is any subset of E that is closed under affine combinations.

Let K_1 and K_2 be convex sets in linear spaces E_1 and E_2, respectively. A mapping $f : K_1 \to K_2$ is said to be *affine* if

$$f(\alpha_1 x_1 + \dots + \alpha_n x_n) = \alpha_1 f(x_1) + \dots + \alpha_n f(x_n),$$

for every convex combination $\alpha_1 x_1 + \dots + \alpha_n x_n \in K_1$. If f is a bijection, then the inverse mapping f^{-1} is automatically also affine and we call f an *affine isomorphism* of K_1 and K_2.

A *convex cone* in a linear space E is a subset C of E such that

(i) $0 \in C$,

(ii) if $\alpha_1, \alpha_2 \geq 0$ and $x_1, x_2 \in C$, then $\alpha_1 x_1 + \alpha_2 x_2 \in C$.

A *base* for a convex cone C is any convex subset K of C such that every non-zero element $y \in C$ may be uniquely expressed as $y = \alpha x$ for some $\alpha \geq 0$ and some $x \in K$. The next criterion is useful when deciding whether a given convex set can be a base for some convex cone.

Proposition 2 (Proposition 10.2 in [12])**.** *Let K be a nonempty convex subset of a linear space E and*

$$C = \{\alpha x \mid \alpha \geq 0,\ x \in K\}.$$

Then C is a convex cone in E and the following conditions are equivalent:

(i) K is a base for C.

(ii) K is contained in an affine subspace A of E such that $0 \notin A$.

A convex cone C is said to be *pointed* provided $x, -x \in C$ implies $x = 0$. Observe that a convex cone possessing a base must be pointed.

If C is a pointed convex cone in a linear space E, then a binary relation $\leq_C$ on E defined by

$$x \leq_C y \ \text{ whenever } y - x \in C,$$

for every $x, y \in E$, makes E into a *partially ordered linear space.* Precisely, this means that for all $x, y, z \in E$ with $x \leq_C y$ and any $\alpha \geq 0$, it follows that

$$x + z \leq_C y + z \quad \text{and} \quad \alpha x \leq_C \alpha y.$$

Moreover, we obtain $C = \{x \in E \mid 0 \leq_C x\}$. A *lattice cone* is any pointed convex cone C in E such that the set C endowed with a partial order $\leq_C$ is a lattice. A *simplex* in a linear space E is any convex subset S of E that is affinely isomorphic to a base for a lattice cone in some linear space. The extreme boundary of a simplex is an affinely independent subset of E [12, Corollary 10.8].

Example 1. *Let $E = \mathbb{R}^k$ and S be the convex hull of a finite affinely independent subset $\{x_1, \dots, x_n\}$ of E. For example, in $\mathbb{R}^3$ we can take the three standard unit basis vectors. We call the set S an $(n-1)$-*simplex. *Every $(n-1)$-simplex is a simplex in the sense of the above definition.*

In the rest of this section E denotes a locally convex Hausdorff space. Compact convex subsets of such a space permit a neat characterization.

Theorem 1 (Krein-Milman)**.** *If K is a compact convex subset of a locally convex space X, then the convex hull of* $\operatorname{ext} K$ *is dense in K.*

Krein-Milman theorem applies to the sets of belief functions studied in both [7] and [8] since they are compact in the product topology of some power of $\mathbb{R}$—see Section IV for more details. On the contrary, the set of all belief measures studied in this paper is not compact in general. The absence of compactness should not come as surprise at all, since already the set of all Borel probability measures fails to be compact in the weak topology, which is considered to be the "right" one for most of applications of topological measure theory to probability and statistics.

III. Space of Belief Measures

We will investigate the space of belief measures by looking at the properties of the corresponding space of certain Borel measures whose support is included in H. Let $\mathbf{M}^{[0,1]}(H)$ be the set of all nonnegative Borel measures μ on H satisfying the condition $\mu(H) \in [0,1]$ and $\mathbf{M}^1(H)$ be the set of all Borel probability measures on H. Observe that

$$\mathbf{M}^{[0,1]}(H) = \{\alpha P \mid \alpha \in [0,1], P \in \mathbf{M}^1(H)\}.$$

Given $(x, y) \in H$, put

$$\varepsilon_{(x,y)}(A) = \begin{cases} 1, & (x,y) \in A, \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $\varepsilon_{(x,y)}$ is a Borel probability measure, which is called the *Dirac measure* at (x, y).

Let f be a function $H \to \mathbb{R}$. The set

$$\operatorname{supp} f = \operatorname{cl} \{(x, y) \in H \mid f(x, y) \neq 0\}$$

is called the *support* of f. Let $\mathcal{C}_c(H)$ be the Banach space with the supremum norm of all continuous functions $H \to \mathbb{R}$ with compact support. We always consider the set $\mathbf{M}^{[0,1]}(H)$ to be endowed with the *vague topology* [13] that is generated by a basis consisting of the sets

$$\left\{P \in \mathbf{M}^{[0,1]}(H) \mid |\textstyle\int_H f_i \, \mathrm{d}P - \int_H f_i \, \mathrm{d}P_0| < \varepsilon,\ i = 1, \dots, n\right\},$$

for every $P_0 \in \mathbf{M}^{[0,1]}(H)$, $f_1, \dots, f_n \in \mathcal{C}_c(H)$ and $\varepsilon > 0$. In this topology, a sequence (P_n) in $\mathbf{M}^{[0,1]}(H)$ vaguely converges to a Borel measure $P \in \mathbf{M}^{[0,1]}(H)$ iff

$$\lim_{n \to \infty} \int_H f \, \mathrm{d}P_n = \int_H f \, \mathrm{d}P$$

for every $f \in \mathcal{C}_c(H)$.

Let $\mathbf{Bel}$ be the set of all belief functions on $\mathcal{B}(\mathbb{R})$. By $\mathbf{Bel}^{[0,1]}$ we denote the image of $\mathbf{M}^{[0,1]}(H)$ via the correspondence as in (1). Precisely, the elements of $\mathbf{Bel}^{[0,1]}$ are exactly the set functions Bel such that

$$\operatorname{Bel}(A) = P[[X, Y] \subseteq A], \quad A \in \mathcal{B}(\mathbb{R}), \tag{3}$$

where $P \in \mathbf{M}^{[0,1]}(H)$. One way to define a topology on $\mathbf{Bel}^{[0,1]}$ is to introduce on it the *vague topology* of capacities [10] generated by the subbase consisting of all the sets of the form

$$\left\{\operatorname{Bel} \in \mathbf{Bel}^{[0,1]} \mid \operatorname{Bel}(K) < \varepsilon\right\},\ \left\{\operatorname{Bel} \in \mathbf{Bel}^{[0,1]} \mid \operatorname{Bel}(G) > \varepsilon\right\},$$

for every $K \in \mathcal{K}, G \in \mathcal{G}$ and $\varepsilon > 0$. The role of the set $\mathbf{Bel}^{[0,1]}$ of "generalized belief measures" is rather technical in this paper as it enables using properties of continuous bijective mappings between two compact spaces in the second part of the proof of Theorem 2.

Proposition 3. *The set of all belief functions* $\mathbf{Bel}$ *on $\mathcal{B}(\mathbb{R})$ is affinely homeomorphic to* $\mathbf{M}^1(H)$.

Proof: Our goal is to find an affine isomorphism of $\mathbf{Bel}$ and $\mathbf{M}^1(H)$ that is simultaneously a homeomorphism. To this end, let

$$a : P \in \mathbf{M}^{[0,1]}(H) \mapsto \operatorname{Bel}, \tag{4}$$

where Bel is given by (3). Since $a(\mathbf{M}^1(H)) = \mathbf{Bel}$, we need only show that the mapping a is an affine homeomorphism from $\mathbf{M}^{[0,1]}(H)$ to $\mathbf{Bel}^{[0,1]}$.

Let $P_1, P_2 \in \mathbf{M}^{[0,1]}(H)$ and $\alpha \in [0,1]$. Since for every $A \in \mathcal{B}(\mathbb{R})$,

$$a(\alpha P_1 + (1-\alpha)P_2)(A) = (\alpha P_1 + (1-\alpha)P_2)[[X,Y] \subseteq A] = \alpha a(P_1)(A) + (1-\alpha)a(P_2)(A),$$

the mapping a is indeed affine. Let $P_1, P_2 \in \mathbf{M}^{[0,1]}(H)$, where $P_1 \neq P_2$. This means that there must be some $(a,b) \in H$ such that

$$P_1(\{(x,y) \in H \mid x \geq a, y \leq b\}) \neq P_2(\{(x,y) \in H \mid x \geq a, y \leq b\}),$$

which reads as

$$P_1[[X,Y] \subseteq [a,b]] \neq P_2[[X,Y] \subseteq [a,b]].$$

The laste inequality is exactly

$$a(P_1)([a,b]) \neq a(P_2)([a,b]).$$

Hence a is affine, injective and onto $\mathbf{Bel}^{[0,1]}$, so it is an affine isomorphism of $\mathbf{M}^{[0,1]}(H)$ and $\mathbf{Bel}^{[0,1]}$.

We will show that a is continuous. It suffices to check sequential continuity as $\mathbf{M}^{[0,1]}(H)$ is a Polish space in the vague topology [13, Theorem 31.5] and $\mathbf{Bel}^{[0,1]}$ is metrizable in the vague topology of capacities [10, p.23]. Suppose that a sequence (P_n) in $\mathbf{M}^{[0,1]}(H)$ vaguely converges to a measure $P \in \mathbf{M}^{[0,1]}(H)$. This means by the first implication in (the proof of) [13, Theorem 30.2] that

$$\limsup_{n\to\infty} P_n(K) \leq P(K), \tag{5}$$

for every compact set $K \subseteq H$, and

$$\liminf_{n\to\infty} P_n(G) \geq P(G), \tag{6}$$

for every open set $G \subseteq H$. By [10, p.22-23] the sequence $(a(P_n))$ vaguely converges to $a(P)$ in $\mathbf{Bel}^{[0,1]}$ iff

$$\limsup_{n\to\infty} a(P_n)(K) \leq a(P)(K), \quad K \in \mathcal{K}, \tag{7}$$

$$\liminf_{n\to\infty} a(P_n)(G) \geq a(P)(G), \quad G \in \mathcal{G}. \tag{8}$$

Let $K \in \mathcal{K}$. Then (5) and compactness of

$$\{(x,y) \in H \mid [x,y] \subseteq K\}$$

in H yield

$$\limsup_{n\to\infty} a(P_n)(K) = \limsup_{n\to\infty} P_n[[X,Y] \subseteq K] = \limsup_{n\to\infty} P_n\big(\{(x,y) \in H \mid [x,y] \subseteq K\}\big) \leq P\big(\{(x,y) \in H \mid [x,y] \subseteq K\}\big) = a(P)(K),$$

which proves (7). The inequality (8) is proven completely analogously by employing (6) together with the openness of

$$\{(x,y) \in H \mid [x,y] \subseteq G\}, \quad G \in \mathcal{G}$$

in H. Thus a is continuous and since $\mathbf{M}^{[0,1]}(H)$ is vaguely compact [13, Corollary 31.3], the set $\mathbf{Bel}^{[0,1]}$ is also vaguely compact. This gives together with injectivity of a that a^{-1} is continuous. Thus a is a homeomorphism of $\mathbf{M}^{[0,1]}(H)$ and $\mathbf{Bel}^{[0,1]}$ (see [14, Lemma I.5.8]). ■

Theorem 2. *The set of all belief function* $\mathbf{Bel}$ *is a metrizable simplex whose extreme boundary is formed by the belief functions*

$$\mathrm{Bel}_{[x,y]}(A) = \begin{cases} 1, & [x,y] \subseteq A, \\ 0, & \textit{otherwise}, \end{cases} \quad A \in \mathcal{B}(\mathbb{R}),$$

for every $(x,y) \in H$. *Moreover, the extreme boundary of* $\mathbf{Bel}$ *is homeomorphic to* H.

Proof: The set of all belief function $\mathbf{Bel}$ is metrizable as a subset of the metrizable space $\mathbf{Bel}^{[0,1]}$. Since $\mathbf{Bel}$ is affinely isomorphic to $\mathbf{M}^1(H)$ by Proposition 3, it suffices to verify that $\mathbf{M}^1(H)$ is a simplex. The convex set $\mathbf{M}^1(H)$ is a base for the set

$$\mathbf{M}^+(H) = \{\alpha P \mid \alpha \in [0,\infty), P \in \mathbf{M}^1(H)\}$$

of all finite nonnegative Borel measures on H due to Proposition 2. The pointed convex cone $\mathbf{M}^+(H)$ generates the usual setwise ordering $\leq$ on the set $\mathbf{M}(H)$ of real-valued Borel measures on H, that is,

$$\mu \leq \nu \text{ whenever } \mu(A) \leq \nu(A), \quad A \in \mathcal{B}(H),$$

where $\mu, \nu \in \mathbf{M}(H)$. Then [15, Proposition 5, p.179] yields that the ordering $\leq$ makes $\mathbf{M}^+(H)$ into a lattice cone in which the infimum is given by

$$(\mu \wedge \nu)(A) = \inf\{\mu(A_1) + \nu(A_2) \mid (A_1, A_2) \text{ partition of } A\},$$

where (A_1, A_2) being a partition of $A \in \mathcal{B}(H)$ means that $A_1, A_2 \in \mathcal{B}(H)$ are disjoint with $A_1 \cup A_2 = A$. The supremum is then

$$(\mu \vee \nu)(A) = -(-\mu \wedge -\nu)(A), \quad A \in \mathcal{B}(H).$$

Since we have shown that $\mathbf{M}^1(H)$ is a base for the lattice cone $\mathbf{M}^+(H)$ in $\mathbf{M}(H)$, it is indeed a simplex.

The extreme boundary of $\mathbf{M}^1(H)$ is exactly the set

$$\{\varepsilon_{(x,y)} \mid (x,y) \in H\}$$

of all Dirac measures on H [16, Proposition 437P]. Proposition 3 yields that the extreme boundary of $\mathbf{Bel}$ is precisely the set

$$\{a(\varepsilon_{(x,y)}) \mid (x,y) \in H\},$$

where a is given by (4). Hence we can conclude that for every $A \in \mathcal{B}(\mathbb{R})$,

$$a(\varepsilon_{(x,y)})(A) = \varepsilon_{(x,y)}[[X,Y] \subseteq A] = \varepsilon_{(x,y)}(\{x',y') \in H \mid [x,y] \subseteq A\}) = \mathrm{Bel}_{[x,y]}(A).$$

The last assertion of the theorem follows from the fact that the mapping

$$(x,y) \in H \mapsto \mathrm{Bel}_{[x,y]}$$

is a composition of the two mappings

$$(x,y) \in H \mapsto \varepsilon_{(x,y)} \text{ and } \varepsilon_{(x,y)} \mapsto \mathrm{Bel}_{[x,y]}\,.$$

The former is a homeomorphism of H and $\operatorname{ext}\mathbf{M}^1(H)$ by [13, p.237] and the latter is homeomorphism of $\operatorname{ext}\mathbf{M}^1(H)$ and $\operatorname{ext}\mathbf{Bel}$ by Proposition 3. ■

The set $\mathbf{Bel}$ is not closed and thus not compact either. It is enough to take a sequence of belief measures $(\mathrm{Bel}_{[0,n]})$. This sequence vaguely converges to the zero belief function in $\mathbf{Bel}^{[0,1]}$: since

$$\mathrm{Bel}_{[0,n]} = a(\varepsilon_{(0,n)})$$

for each $n \in \mathbb{N}$, and the sequence $(\varepsilon_{(0,n)})$ vaguely converges to the zero measure as

$$\lim_{n\to\infty} \int_H f \,\mathrm{d}\varepsilon_{(0,n)} = \lim_{n\to\infty} f(0,n) = 0,$$

for every $f \in \mathcal{C}_c(H)$, we obtain from continuity of a that $a(0) = 0$.

IV. Comparison with Other Spaces of Belief Functions

The study of various spaces of belief functions has already appeared in a number of papers [7], [8], [17]. However, our results neither imply nor are implied by the results of Shafer [7] and Denneberg with Brüning [8]. The basic setting for belief functions developed in those two papers is as follows. Let Ω be a nonempty set and $\mathcal{A}$ be an algebra of subsets of Ω. We say that $\mathrm{Bel} : \mathcal{A} \to [0,1]$ is a *belief measure* on the algebra $\mathcal{A}$ if it satisfies the following conditions:

(i) $\mathrm{Bel}(\emptyset) = 0, \mathrm{Bel}(\Omega) = 1$

(ii) if $A, B \in \mathcal{A}$ are such that $A \subseteq B$, then

$$\mathrm{Bel}(A) \le \mathrm{Bel}(B)$$

(iii) if $n \ge 2$ and $A_1, \dots, A_n \in \mathcal{A}$, then

$$\mathrm{Bel}\Big(\bigcup_{i=1}^n A_i\Big) + \sum_{\substack{I \subseteq \{1,\dots,n\} \\ I \neq \emptyset}} (-1)^{|I|}\,\mathrm{Bel}\Big(\bigcap_{i \in I} A_i\Big) \ge 0$$

The set $\mathbf{Bel}_{\mathcal{A}}$ of all such belief measures on $\mathcal{A}$ is then convex and compact in the subspace topology of the product topological space $[0,1]^{\mathcal{A}}$. In fact, it is not difficult to show that $\mathrm{Bel}_{\mathcal{A}}$ is also a simplex. It is known [8, Proposition 4.1] that the set of extreme points of $\mathbf{Bel}_{\mathcal{A}}$ coincides with the set of all $\{0,1\}$-valued belief measures. However, the compactness of $\mathbf{Bel}_{\mathcal{A}}$ must be put in contrast with the lack of compactness of $\mathbf{Bel}$. The framework for belief functions adopted in this paper cannot be seen as a special instance: if $\Omega = \mathbb{R}$, $\mathcal{A} = \mathcal{B}(\mathbb{R})$, then $\mathbf{Bel} \subsetneq \mathbf{Bel}_{\mathcal{A}}$. The inclusion is proper since there exist elements in $\mathbf{Bel}_{\mathcal{A}}$ that fail to satisfy some of the conditions (iv)-(vi) in Proposition 1.

On the other hand, the space of belief functions on the algebra of all subsets of a finite universe Ω can be identified with a particular subset of $\mathbf{Bel}$ in our setting. Precisely, let $|\Omega| = n$, where $n > 0$, and $\mathcal{A} = 2^\Omega$. Then $\mathbf{Bel}_{\mathcal{A}}$ is an $(2^n - 2)$-simplex [17, Corollary 1]. In order to establish a link to the belief measures on real numbers, put $N = 2^n - 1$ and consider a finite subset C of H with N elements. The set of all probability measures on 2^C is an $(N-1)$-simplex that can be identified with the set of all mappings $p : C \to [0,1]$ such that $\sum_{(x,y)\in C} p(x,y) = 1$. Each such mapping corresponds to a Borel probability measure P supported by a subset of C, which defines in turn a belief measure on $\mathcal{B}(\mathbb{R})$ by using (1): for every $A \in \mathcal{B}(\mathbb{R})$, we get

$$\mathrm{Bel}(A) = P(\{(x,y) \in C \mid [x,y] \subseteq A\}) = \sum_{\substack{(x,y)\in C \\ [x,y]\subseteq A}} p(x,y).$$

An inessential modification of the proofs in this article shows that the set of all belief functions whose probability distributions P are supported by a subset of C is an $(N-1)$-simplex. This makes possible to identify such belief measures with the set $\mathbf{Bel}_{\mathcal{A}}$ of all belief funtions on $\mathcal{A}$.

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References

[1] A. P. Dempster, “Upper and lower probabilities induced by a multivalued mapping,” *Ann. Math. Statist.*, vol. 38, pp. 325–339, 1967.

[2] ——, “Upper and lower probabilities generated by a random closed interval.” *Ann. Math. Statist.*, vol. 39, pp. 957–966, 1968.

[3] G. Shafer, *A Mathematical Theory of Evidence*, ser. Princeton University Press. NJ: Princeton, 1976.

[4] P. Smets, “Belief functions on real numbers,” *Internat. J. Approx. Reason.*, vol. 40, pp. 181–223, 2005.

[5] A. Aregui and T. Denœux, “Constructing consonant belief functions from sample data using confidence sets of pignistic probabilities,” *Internat. J. Approx. Reason.*, vol. 49, no. 3, pp. 575–594, 2008.

[6] T. Denœux, “Extending stochastic ordering to belief functions on the real line,” *Inform. Sci.*, vol. 179, pp. 1362–1376, 2009.

[7] G. Shafer, “Allocations of probability,” *Ann. Probab.*, vol. 7, no. 5, pp. 827–839, 1979.

[8] M. Brüning and D. Denneberg, “The extreme points of the set of belief measures,” *Internat. J. Approx. Reason.*, vol. 48, no. 3, pp. 670–675, 2008.

[9] G. Choquet, “Theory of capacities,” *Ann. Inst. Fourier, Grenoble*, vol. 5, pp. 131–295 (1955), 1953–1954.

[10] G. L. O'Brien, “Sequences of capacities, with connections to large-deviation theory,” *J. Theoret. Probab.*, vol. 9, no. 1, pp. 19–35, 1996.

[11] I. Molchanov, *Theory of random sets*, ser. Probability and its Applications (New York). London: Springer-Verlag London Ltd., 2005.

[12] K. R. Goodearl, *Partially ordered abelian groups with interpolation*, ser. Mathematical Surveys and Monographs. Providence, RI: American Mathematical Society, 1986, vol. 20.

[13] H. Bauer, *Maß- und Integrationstheorie*, 2nd ed., ser. de Gruyter Lehrbuch. [de Gruyter Textbook]. Berlin: Walter de Gruyter & Co., 1992.

[14] N. Dunford and J. T. Schwartz, *Linear Operators. I. General Theory*, ser. With the assistance of W. G. Bade and R. G. Bartle. Pure and Applied Mathematics, Vol. 7. Interscience Publishers, Inc., New York, 1958.

[15] M. M. Rao, *Measure theory and integration*, ser. Pure and Applied Mathematics (New York). New York: John Wiley & Sons Inc., 1987, a Wiley-Interscience Publication.

[16] D. H. Fremlin, *Measure theory. Vol. 4*. Torres Fremlin, Colchester, 2006, topological measure spaces. Part I, II, Corrected second printing of the 2003 original.

[17] F. Cuzzolin, “A geometric approach to the theory of evidence,” *IEEE Transactions on Systems, Man, and Cybernetics part C*, 2007.